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On classification of second-order differential equations with complex coefficients ☆

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ABSTRACT

This paper is concerned with the deficiency index problem of second-order differential equations with complex coefficients. It is known that this class of equations is classified into cases I, II, and III according to the number of linearly independent solutions in suitable weighted square integrable spaces. In this study, the original equation is reformulated into a new formally self-adjoint differential system by introducing a new spectral parameter and the relationship between the classifications of the equation and the system is obtained. Moreover, the exact dependence of cases II and III on the corresponding half planes is given and some criteria of the three cases are established.

1. Introduction

Consider the Sturm–Liouville differential equation with complex coefficients

$$-\big(p(t)y'(t)\big)' + q(t)y(t) = \lambda w(t)y(t), \quad t \in [a,b), \tag{1.1}$$

where $-\infty < a < b \leqslant +\infty$, p and q are complex-valued functions, w is a weight function, $p(t) \neq 0$ and $w(t) > 0$, a.e. $t \in [a,b)$, $p^{-1}(t)$, $q(t)$, and $w(t)$ are locally integrable on $[a,b)$, λ is a spectral parameter.

Since Eq. (1.1) is formally self-adjoint if and only if $p(t)$ and $q(t)$ are real, Eq. (1.1) is called to be formally non-self-adjoint when $\operatorname{Im} p(t) \neq 0$ or $\operatorname{Im} q(t) \neq 0$. The formally self-adjoint Eq. (1.1) can be divided into two cases: the limit point and limit circle cases. H. Weyl in 1910 first gave this dichotomy for singular formally self-adjoint second-order linear differential equations [20]. His work has been greatly developed and generalized to higher-order differential equations and Hamiltonian systems and some limit point and limit circle criteria for higher-order differential equations and Hamiltonian systems were formulated (cf. [1,2,10–17,19]). Sims obtained an extension of the Weyl's classification to a class of formally non-self-adjoint second-order linear differential equations in 1957 [18]. Brown et al. extended this work to Eq. (1.1) [4]. They divided Eq. (1.1) into three cases I, II, and III by using the m-functions which were defined on a collection of rotated half planes (see Section 2 for the detailed definition of the rotated half planes). This classification is related to the rotated half planes compared with the classification given by Sims. So, it is not simply a straightforward generalization of the classification given by Sims. In addition, they also gave examples for all three cases by using methods of asymptotic analysis in [4]. Recently, Bennewitz and Brown constructed examples for all three cases by the generalization of the classical Weyl limit point and limit circle analysis [3]. It was not known in [4] whether there exists the case where a differential equation is in the case II with respect to (briefly, w.r.t.) a rotated half plane and case III w.r.t. another one, i.e., whether cases II and III depend on the

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corresponding half planes [4, Remark 2.4]. More recently, Jing proved that there indeed exists such a case by illustrating examples [9]. For more relevant results, the reader refers to [5–8].

It is noted that how cases II and III depend on the corresponding half planes has not been considered up to now and the case of Eq. (1.1) is very important to study the spectral problems of an operator generated by Eq. (1.1) [4]. In the present paper, we present the exact dependence of cases II and III on the corresponding half planes. We reformulate Eq. (1.1) into a new formally self-adjoint differential system by introducing a new spectral parameter and give the relationship between the classifications of Eq. (1.1) and the new system. Using this relationship we set up some criteria of the three cases given in [4].

The rest of this present paper is organized as follows. In Section 2, we introduce the classification of Eq. (1.1) and give the exact dependence of cases II and III on the corresponding half planes. In Section 3, the relationship between the classification of Eq. (1.1) and that of Hamiltonian systems is given. Several criteria of the cases I, II, and III are established in Section 4.

2. Classification of Eq. (1.1)

In this section, we introduce the classification of Eq. (1.1) given by Brown et al. in [4] and show how cases II and III of this classification depend on the corresponding half planes.

The endpoint a is called to be *regular* if $-\infty < a < +\infty$ and $\int_a^c (w(t) + |p(t)|^{-1} + |q(t)|)\,\mathrm{d}t < \infty$ for some $c \in (a,b)$. Otherwise, it is called to be *singular* (cf. [4]). The regularity or singularity of the endpoint b is defined in the similar way. Throughout the present paper we focus on the case where a is regular and b is singular.

To begin with, let us introduce the Hilbert space

$$L_w^2 := \left\{ y\colon \int_a^b w(t)\big|y(t)\big|^2\,\mathrm{d}t < \infty \right\}$$

with the inner product $\langle y, z\rangle := \int_a^b \bar{z}(t) w(t) y(t)\,\mathrm{d}t$ and the norm $\|y\| = (\langle y, y\rangle)^{1/2}$ for $y, z \in L_w^2$. If $y \in L_w^2$, then y is called to be square integrable.

Now, it is assumed throughout the present paper that

$$\Omega := \overline{\mathrm{co}}\left\{ \frac{q(t)}{w(t)} + rp(t),\ t \in [a,b),\ 0 < r < \infty \right\} \neq \mathbb{C}, \tag{2.1}$$

where $\overline{\mathrm{co}}$ denotes the closed convex hull (i.e., the smallest closed convex set containing the exhibited set). Then, by the relevant results about supporting lines in convex analysis, it holds that for each point on the boundary $\partial\Omega$, there exists a line through this point such that every point of Ω either lies in the same side of this line or is on it, that is, there exists a supporting line through this point. Let K be a point on $\partial\Omega$. Denote by L an arbitrary supporting line touching Ω at K, which may be the tangent to Ω at K if it exists. We then perform a transformation of the complex plane $z \mapsto z - K$ and a rotation through an angle $\theta \in (-\pi, \pi]$, so that the image of L now coincides with the imaginary axis and the set Ω lies in the nonnegative half plane. Therefore, for all $t \in [a,b)$ and $0 < r < \infty$,

$$\mathrm{Re}\left\{ \left[\frac{q(t)}{w(t)} + rp(t) - K \right] e^{i\theta} \right\} \geqslant 0. \tag{2.2}$$

For such admissible values of K and θ, the negative rotated half plane can be expressed as

$$\Lambda_{\theta,K} = \left\{ \lambda \in \mathbb{C}\colon \mathrm{Re}\big[(\lambda - K)e^{i\theta}\big] < 0 \right\}.$$

Clearly, for all $\lambda \in \Lambda_{\theta,K}$, we have

$$\mathrm{Re}\big[(\lambda - K)e^{i\theta}\big] \leqslant -\delta < 0, \tag{2.3}$$

where $\delta = \delta_{\theta,K}(\lambda)$ is the distance from λ to the boundary $\partial\Lambda_{\theta,K}$. Here, we remark that the angle of rotation θ may not be unique for the same point $K \in \partial\Omega$ since there may exist many supporting lines through K. Set

$$S := \left\{ (\theta, K)\colon K \in \partial\Omega,\ \mathrm{Re}\big[(z - K)e^{i\theta}\big] \geqslant 0 \text{ for all } z \in \Omega \right\}.$$

Then S consists of all the admissible values of K and θ.

Note that (2.2) holds for all $0 < r < \infty$ for a given $(\theta, K) \in S$. Then, by setting $r \to 0$, it can be obtained from (2.2) that

$$\mathrm{Re}\big[\big(q(t) - Kw(t)\big)e^{i\theta}\big] \geqslant 0, \quad t \in [a,b). \tag{2.4}$$

Further, we get from (2.2) that for $0 < r < \infty$,

$$\mathrm{Re}\left\{ \left[\frac{q(t)}{rw(t)} + p(t) - \frac{K}{r} \right] e^{i\theta} \right\} \geqslant 0, \quad t \in [a,b),$$

and hence by letting $r \to \infty$, it follows that

$$\mathrm{Re}\big[p(t)e^{i\theta}\big] \geqslant 0, \quad t \in [a,b). \tag{2.5}$$

Let $(\theta, K) \in S$ and $\Lambda_{\theta,K}$ be the corresponding half plane. Using a nesting circle method, Brown et al. proved that Eq. (1.1) can be classified into three cases w.r.t. the half plane $\Lambda_{\theta,K}$ [4, Theorem 2.1]. Each of them is distinct.

Definition 2.1. (See [4, Theorem 2.1].) Let $(\theta, K) \in S$. Then for $\lambda \in \Lambda_{\theta,K}$,

(1) if Eq. (1.1) has exactly one linearly independent solution satisfying

$$\int_a^b \mathrm{Re}\big(e^{i\theta} p\big)\big|y'\big|^2\,\mathrm{d}t + \int_a^b \mathrm{Re}\big[e^{i\theta}(q - Kw)\big]|y|^2\,\mathrm{d}t + \int_a^b w|y|^2\,\mathrm{d}t < \infty \tag{2.6}$$

and this is the only linearly independent solution of Eq. (1.1) in L^2_w, then Eq. (1.1) is called to be in the case I;
(2) if Eq. (1.1) has exactly one linearly independent solution satisfying (2.6), but all the solutions of Eq. (1.1) are in L^2_w, then Eq. (1.1) is called to be in the case II;
(3) if all the solutions of Eq. (1.1) satisfy (2.6) and hence are in L^2_w, then Eq. (1.1) is called to be in the case III.

Remark 2.1. (See [4, Remark 2.2].) By the method of variation of parameters, it can be concluded that this classification w.r.t. $\Lambda_{\theta,K}$ is independent of λ in the sense that

(i) if all the solutions of Eq. (1.1) satisfy (2.6) for some $\lambda_0 \in \Lambda_{\theta,K}$ (i.e., case III), then the same is true for all $\lambda \in \mathbb{C}$;
(ii) if all the solutions of Eq. (1.1) are in L^2_w for some $\lambda_0 \in \mathbb{C}$, then the same is true for all $\lambda \in \mathbb{C}$.

Remark 2.2.

(i) Eq. (1.1) is in the case I if it has a solution y not to be in L^2_w. In fact, it can be concluded from $y \notin L^2_w$, (2.4), and (2.5) that y does not satisfy (2.6);
(ii) By (i) of this remark and (ii) of Remark 2.1, if Eq. (1.1) is in the case I w.r.t. some Λ_{θ_0,K_0}, then the same is true for all $\Lambda_{\theta,K}$, that is, case I is independent of $\Lambda_{\theta,K}$;
(iii) If p and q are real and Eq. (1.1) is in the case I, then Eq. (1.1) is in the limit point case of the Weyl's classification.

It has been known that case I is independent of $\Lambda_{\theta,K}$ by (ii) of Remark 2.2. However, cases II and III are dependent on $\Lambda_{\theta,K}$ in general, that is, there exist Λ_{θ_1,K_1} and Λ_{θ_2,K_2} such that (1.1) is in the case II w.r.t. Λ_{θ_1,K_1} and case III w.r.t. Λ_{θ_2,K_2}, by [9, Examples 3.1 and 3.2]. Now, we give the exact dependence of cases II and III on the corresponding half planes by the following theorem:

Theorem 2.1. *Let*

$$\mathcal{A} := \big\{\theta\colon \text{ there exists } K \in \partial\Omega \text{ such that } (\theta, K) \in S\big\}.$$

Assume that $\theta_1, \theta_2 \in \mathcal{A}$ and $\theta_2 \neq \theta_1 \pmod{\pi}$. If Eq. (1.1) is in the case III w.r.t. Λ_{θ_1,K_1} and Λ_{θ_2,K_2}, respectively, then Eq. (1.1) is in the case III w.r.t. all $\Lambda_{\theta,K}$.

Proof. Let $p(t) = |p(t)|e^{i\phi(t)}$. Then we have

$$\mathrm{Re}\big(e^{i\theta_l} p(t)\big) = \big|p(t)\big|\cos\big(\theta_l + \phi(t)\big), \quad l = 1, 2. \tag{2.7}$$

Using the formulae

$$\sin(\theta_2 - \theta_1) = \sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1,$$
$$\cos\big(\theta_l + \phi(t)\big) = \cos\theta_l\cos\phi(t) - \sin\theta_l\sin\phi(t)$$

for $l = 1, 2$, and noting that $\sin(\theta_2 - \theta_1) \neq 0$ by $\theta_2 \neq \theta_1 \pmod{\pi}$, we have

$$\cos\phi(t) = \frac{\sin\theta_2}{\sin(\theta_2 - \theta_1)}\cos\big(\theta_1 + \phi(t)\big) - \frac{\sin\theta_1}{\sin(\theta_2 - \theta_1)}\cos\big(\theta_2 + \phi(t)\big),$$
$$\sin\phi(t) = \frac{\cos\theta_2}{\sin(\theta_2 - \theta_1)}\cos\big(\theta_1 + \phi(t)\big) - \frac{\cos\theta_1}{\sin(\theta_2 - \theta_1)}\cos\big(\theta_2 + \phi(t)\big). \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\begin{aligned}
\operatorname{Re} p(t) &= \frac{\sin\theta_2}{\sin(\theta_2-\theta_1)}\operatorname{Re}\big(e^{i\theta_1}p(t)\big) - \frac{\sin\theta_1}{\sin(\theta_2-\theta_1)}\operatorname{Re}\big(e^{i\theta_2}p(t)\big),\\
\operatorname{Im} p(t) &= \frac{\cos\theta_2}{\sin(\theta_2-\theta_1)}\operatorname{Re}\big(e^{i\theta_1}p(t)\big) - \frac{\cos\theta_1}{\sin(\theta_2-\theta_1)}\operatorname{Re}\big(e^{i\theta_2}p(t)\big).
\end{aligned} \tag{2.9}$$

Hence, it follows from (2.5) and (2.9) that there exists a positive constant M_0 such that

$$\big|p(t)\big| \leqslant M_0\big[\operatorname{Re}\big(e^{i\theta_1}p(t)\big) + \operatorname{Re}\big(e^{i\theta_2}p(t)\big)\big]. \tag{2.10}$$

With a similar argument, we can prove that there exists a positive constant $\tilde{M}_0$ such that for $\lambda \in \mathbb{C}$,

$$\big|q(t)-\lambda w(t)\big| \leqslant \tilde{M}_0\big\{\big|\operatorname{Re}\big[e^{i\theta_1}\big(q(t)-\lambda w(t)\big]\big| + \big|\operatorname{Re}\big[e^{i\theta_2}\big(q(t)-\lambda w(t)\big]\big|\big\}. \tag{2.11}$$

Now, let $(\theta, K) \in S$ and $\lambda \in \Lambda_{\theta,K}$. Let z be a solution of Eq. (1.1). Suppose that Eq. (1.1) is in the case III w.r.t. Λ_{θ_1,K_1} and Λ_{θ_2,K_2}, respectively. Then, it follows from (i) of Remark 2.1 that z satisfies (2.6) with θ, K replaced by θ_1, K_1 and θ_2, K_2, respectively. Then, we can get from (2.4)–(2.6), and $z \in L^2_w$ that

$$\begin{aligned}
&\int_a^b \operatorname{Re}\big(e^{i\theta_l}p(t)\big)\big|z'(t)\big|^2\,\mathrm{d}t < +\infty,\\
&\int_a^b \big|\operatorname{Re}\big[e^{i\theta_l}\big(q(t)-\lambda w(t)\big)\big]\big|\big|z(t)\big|^2\,\mathrm{d}t < +\infty, \quad l=1,2.
\end{aligned} \tag{2.12}$$

It follows from (2.10)–(2.12) that

$$\begin{aligned}
&\int_a^b \big|p(t)\big|\big|z'(t)\big|^2\,\mathrm{d}t < +\infty,\\
&\int_a^b \big|q(t)-\lambda w(t)\big|\big|z(t)\big|^2\,\mathrm{d}t < +\infty.
\end{aligned} \tag{2.13}$$

Clearly, (2.13) gives that

$$\begin{aligned}
&\int_a^b \operatorname{Re}\big(e^{i\theta}p(t)\big)\big|z'(t)\big|^2\,\mathrm{d}t < +\infty,\\
&\int_a^b \operatorname{Re}\big[e^{i\theta}\big(q(t)-\lambda w(t)\big)\big]\big|z(t)\big|^2\,\mathrm{d}t < +\infty.
\end{aligned} \tag{2.14}$$

Note that $\lambda \in \Lambda_{\theta,K}$. We then get from (2.14) and (2.3) that (2.6) holds for each solution z of Eq. (1.1). Hence, Eq. (1.1) is in the case III w.r.t. $\Lambda_{\theta,K}$. This completes the proof. □

The following result is a direct consequence of Theorem 2.1.

Corollary 2.1. *If Eq.* (1.1) *is in the case* II *w.r.t. an* $\Lambda_{\theta,K}$, *then there exists at most one* $\theta_0 \in \mathcal{A}$ (mod π) *such that Eq.* (1.1) *is in the case* III *w.r.t.* Λ_{θ_0,K_0}.

3. Relationship between classification of (1.1) and that of Hamiltonian systems

In this section we shall study the deficiency index problem of Eq. (1.1). To this end, we transform Eq. (1.1) into a formally self-adjoint Hamiltonian system by introducing a new spectral parameter, and then present the relationship between the classification of Eq. (1.1) given by Definition 2.1 and that of this Hamiltonian system.

In the rest of this section, we choose $(\theta, K) \in S$ and let $\Lambda_{\theta,K}$ be the corresponding rotated half plane. Fix $\lambda \in \Lambda_{\theta,K}$ and set

$$W(t,\lambda,\theta) = \operatorname{diag}\big(w_1(t,\lambda,\theta), w_2(t,\theta)\big), \tag{3.1}$$

where

$$w_1(t,\lambda,\theta) := \operatorname{Re}\big[e^{i\theta}\big(q(t)-\lambda w(t)\big)\big], \qquad w_2(t,\theta) := \big|p(t)\big|^{-2}\operatorname{Re}\big(e^{i\theta}p(t)\big).$$

It follows from (2.3)–(2.5) that $w_1(t,\lambda,\theta) > 0$ and $w_2(t,\theta) \geqslant 0$, a.e. $t \in [a,b)$, that is, $W(t,\lambda,\theta)$ is nonnegative for any given $\lambda \in \Lambda_{\theta,K}$. Now, introduce the space L^2_W, the set of all locally integrable 2-dimensional functions $Y(t)$ such that

$$\int_a^b Y^*(t)W(t,\lambda,\theta)Y(t)\,dt < +\infty.$$

The space L^2_W is a Hilbert space with the inner product $\langle Y, Z\rangle_W = \int_a^b Z^*(t)W(t,\lambda,\theta)Y(t)\,dt$, where $Y(t) = (y(t), v(t))^T$ with $y(t), v(t) \in \mathbb{C}$ and $Y^*(t)$ is the complex conjugate transpose of $Y(t)$. Denote $\|Y\|_W = (\langle Y, Y\rangle_W)^{1/2}$ for $Y \in L^2_W$. If $Y \in L^2_W$, then Y is called to be square integrable. Here, we remark that if W is singular, L^2_W is a quotient space in the sense that $Y = Z$ if $\|Y - Z\|_W = 0$.

For fixed $\lambda \in \Lambda_{\theta,K}$, let us consider the singular Hamiltonian differential system

$$JY'(t) = \big(P(t,\lambda,\theta) + \xi W(t,\lambda,\theta)\big)Y(t), \quad t \in [a,b), \tag{3.2}$$

with a new spectral parameter ξ, where $W(t,\lambda,\theta)$ is defined as in (3.1) and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad P(t,\lambda,\theta) = \begin{pmatrix} -c(t,\lambda,\theta) & 0 \\ 0 & d(t,\theta) \end{pmatrix},$$

$$c(t,\lambda,\theta) := \operatorname{Im}\big[e^{i\theta}\big(q(t) - \lambda w(t)\big)\big], \qquad d(t,\theta) := \big|p(t)\big|^{-2}\operatorname{Im}\big(e^{i\theta}p(t)\big).$$

For this system the standard classification of limit point and limit circle cases states that *if for some $\xi_0 \in \mathbb{C}$, all the solutions of system* (3.2) *are in L^2_W, then system* (3.2) *is called to be in the limit circle case* (*briefly, l.c.c.*) *at $t = b$. Otherwise, system* (3.2) *is called to be in the limit point case* (*briefly, l.p.c.*) *at $t = b$* (cf., e.g., [14]). This classification is independent of $\xi \in \mathbb{C}$ by Corollary 3.1 below and system (3.2) has at least one linearly independent square integrable solution for each $\xi \in \mathbb{C}$ with $\operatorname{Im}\xi \neq 0$ by [11, Theorem 5.4]. The following result gives the relationship between the solutions of Eq. (1.1) and system (3.2) and plays an important role in this present paper.

Lemma 3.1. *If $y(t)$ is a solution of Eq.* (1.1), *then $Y(t) = (y(t), v(t))^T$ with $v(t) = -ie^{i\theta}p(t)y'(t)$ is a solution of system* (3.2) *with $\xi = i$, that is, $Y(t)$ satisfies the system*

$$JY'(t) = \big(P(t,\lambda,\theta) + iW(t,\lambda,\theta)\big)Y(t), \quad t \in [a,b). \tag{3.3}$$

Conversely, if $Y(t) = (y(t), v(t))^T$ is a solution of system (3.3), *then the first component $y(t)$ is a solution of Eq.* (1.1) *and $v(t) = -ie^{i\theta}p(t)y'(t)$.*

Proof. Set

$$v(t) = -ie^{i\theta}p(t)y'(t). \tag{3.4}$$

It follows that

$$y'(t) = i\big|p(t)\big|^{-2}\overline{\big(e^{i\theta}p(t)\big)}v(t) = \big(d(t,\theta) + iw_2(t,\theta)\big)v(t). \tag{3.5}$$

Inserting (3.4) into (1.1), we get that

$$v'(t) = \big(c(t,\lambda,\theta) - iw_1(t,\lambda,\theta)\big)y(t). \tag{3.6}$$

So, $Y(t) = (y(t), v(t))^T$ with $v(t) = -ie^{i\theta}p(t)y'(t)$ is a solution of system (3.3). Conversely, if $Y(t) = (y(t), v(t))^T$ is a solution of system (3.3), then it satisfies (3.5) and (3.6). From (3.5), we get that (3.4) holds. Further, inserting (3.4) into (3.6), we get that (1.1) holds, and hence, the first component $y(t)$ is a solution of Eq. (1.1) and $v(t) = -ie^{i\theta}p(t)y'(t)$. This completes the proof. □

Now, we give the relationship between the classifications of Eq. (1.1) and system (3.2).

Theorem 3.1. *For $(\theta, K) \in S$, system* (3.2) *is in the l.c.c. at $t = b$ if and only if Eq.* (1.1) *is in the case* III *w.r.t. $\Lambda_{\theta,K}$, or equivalently, system* (3.2) *is in the l.p.c. at $t = b$ if and only if Eq.* (1.1) *is not in the case* III, *i.e., either in the case* I *or case* II *w.r.t. $\Lambda_{\theta,K}$.*

Proof. Let $(\theta, K) \in S$ and let $Y(t) = (y(t), v(t))^T$ be a solution of (3.3). Then, from Lemma 3.1, we have

$$\begin{aligned}\int_a^b Y^*(t)W(t,\lambda,\theta)Y(t)\,dt &= \int_a^b \operatorname{Re}\big[e^{i\theta}\big(q(t) - \lambda w(t)\big)\big]\big|y(t)\big|^2\,dt + \int_a^b \big|p(t)\big|^{-2}\operatorname{Re}\big(e^{i\theta}p(t)\big)\big|v(t)\big|^2\,dt \\ &= \int_a^b \operatorname{Re}\big[e^{i\theta}\big(q(t) - \lambda w(t)\big)\big]\big|y(t)\big|^2\,dt + \int_a^b \operatorname{Re}\big(e^{i\theta}p(t)\big)\big|y'(t)\big|^2\,dt,\end{aligned}$$

which yields that the inequality

$$\int_a^b Y^*(t)W(t,\lambda,\theta)Y(t)\,\mathrm{d}t < \infty \tag{3.7}$$

holds if and only if the inequality

$$\int_a^b \mathrm{Re}\big[e^{i\theta}\big(q(t)-\lambda w(t)\big)\big]\big|y(t)\big|^2\,\mathrm{d}t + \int_a^b \mathrm{Re}\big(e^{i\theta}p(t)\big)\big|y'(t)\big|^2\,\mathrm{d}t < \infty \tag{3.8}$$

holds. It follows from (2.3) that (3.8) is equivalent to (2.6). Clearly, if (2.6) holds, then $y \in L^2_w$. Suppose that (3.2) is in the l.c.c. at $t=b$. Then every solution $Y=(y,v)^T$ of (3.3) satisfies (3.7). So, by Lemma 3.1, every solution y of (1.1) satisfies (3.8). Then it satisfies (2.6) and hence is in L^2_w. So, Eq. (1.1) is in the case III w.r.t. $\Lambda_{\theta,K}$. Conversely, if Eq. (1.1) is in the case III w.r.t. $\Lambda_{\theta,K}$, then all the solutions of (1.1) satisfy (2.6), and consequently, satisfy (3.8). Therefore, all the solutions of (3.3) satisfy (3.7) by Lemma 3.1. So, system (3.2) is in the l.c.c. at $t=b$. This completes the proof. □

Remark 3.1. Unlike the method used in [4], applying the above result one can also divide Eq. (1.1) into three cases: the cases I, II, and III (cf. [9]).

Finally, we prepare the following result about the invariant properties of the number of linearly independent square integrable solutions for the systems

$$JY'(t) = \big(Q_1(t) + \xi \tilde{W}(t)\big)Y(t), \quad t \in [a,b), \tag{3.9}$$

$$JY'(t) = \big(Q_2(t) + \xi \tilde{W}(t)\big)Y(t), \quad t \in [a,b), \tag{3.10}$$

where $\tilde{W}(t) = \mathrm{diag}(\tilde{w}_1(t), \tilde{w}_2(t))$ is a weight function with $\tilde{w}_1(t) > 0$ and $\tilde{w}_2(t) \geqslant 0$, a.e. $t \in [a,b)$; $Q_l(t) = \mathrm{diag}(-c_l(t), d_l(t))$; $\tilde{w}_l(t)$, $c_l(t)$, and $d_l(t)$ $(l=1,2)$ are locally integrable functions on $[a,b)$.

Lemma 3.2. *Let $c_l(t)$ and $d_l(t)$ $(l=1,2)$ be complex-valued functions. Assume that there exists a positive constant k such that $|c_2(t)-c_1(t)| \leqslant k\tilde{w}_1(t)$ and $|d_2(t)-d_1(t)| \leqslant k\tilde{w}_2(t)$, $t \in [a,b)$. Then all the solutions of system* (3.9) *are in $L^2_{\tilde{W}}$ if and only if all the solutions of system* (3.10) *are in $L^2_{\tilde{W}}$.*

Since the above result can be proved by the method of variation of parameters and Schwarz inequality, we omit the detailed proof here. The following result is a direct consequence of Lemma 3.2.

Corollary 3.1. *If all the solutions of system* (3.2) *are in L^2_W for some $\xi_0 \in \mathbb{C}$, then the same is true for all $\xi \in \mathbb{C}$.*

4. Criteria of cases I, II, and III

In this section, we establish some criteria for Eq. (1.1) to be in the cases I, II, and III, respectively.

We first establish several criteria for Eq. (1.1) to be in the case I.

Theorem 4.1. *Assume that $\mathcal{A}$ contains at least two different elements θ_1 and θ_2 $(\mathrm{mod}\ \pi)$, where $\mathcal{A}$ is defined as in Theorem* 2.1. *If $\int_a^b \sqrt{\frac{w(t)}{|p(t)|}}\,\mathrm{d}t = \infty$, then Eq.* (1.1) *is in the case* I.

Proof. Since $\theta_1, \theta_2 \in \mathcal{A}$ are different $(\mathrm{mod}\ \pi)$, the imaginary axes of the two half planes Λ_{θ_1,K_1} and Λ_{θ_2,K_2} intersect. Therefore, we have $\Lambda_{\theta_1,K_1} \cap \Lambda_{\theta_2,K_2} \neq \emptyset$. Choose $\lambda \in \Lambda_{\theta_1,K_1} \cap \Lambda_{\theta_2,K_2}$. Let $z(t)$ be the solution of (1.1) satisfying the initial value conditions

$$z(a) = 1, \qquad z'(a) = 0. \tag{4.1}$$

Then we get from (1.1) that

$$\big(p(t)z'(t)\bar{z}(t)\big)' = \big(q(t) - \lambda w(t)\big)\big|z(t)\big|^2 + p(t)\big|z'(t)\big|^2,$$

which, together with (2.3)–(2.5) and (4.1), yields that for $l = 1,2$,

$$\operatorname{Re}\bigl(e^{i\theta_l}p(t)z'(t)\bar{z}(t)\bigr)=\int_a^t \operatorname{Re}\bigl[e^{i\theta_l}\bigl(q(\tau)-\lambda w(\tau)\bigr)\bigr]\bigl|z(\tau)\bigr|^2\,d\tau+\int_a^t \operatorname{Re}\bigl(e^{i\theta_l}p(\tau)\bigr)\bigl|z'(\tau)\bigr|^2\,d\tau$$
$$\geqslant \delta_l\int_a^t w(\tau)\bigl|z(\tau)\bigr|^2\,d\tau, \tag{4.2}$$

where δ_l is the distance from λ to $\partial\Lambda_{\theta_l,K_l}$. It is clear that for $t\in[a,b)$,

$$\bigl|p(t)z'(t)\bar{z}(t)\bigr|=\bigl|e^{i\theta_l}p(t)z'(t)\bar{z}(t)\bigr|\geqslant \operatorname{Re}\bigl(e^{i\theta_l}p(t)z'(t)\bar{z}(t)\bigr),\quad l=1,2. \tag{4.3}$$

Choose $t_1\in(a,b)$ and let $h:=\min\{\delta_1\int_a^{t_1}w(\tau)|z(\tau)|^2\,d\tau,\ \delta_2\int_a^{t_1}w(\tau)|z(\tau)|^2\,d\tau\}$. Then we get from (4.2) and (4.3) that

$$\bigl|p(t)z'(t)\bar{z}(t)\bigr|\geqslant h,\quad t_1\leqslant t<b. \tag{4.4}$$

Note that $\operatorname{Re}[e^{i\theta_l}(q(t)-\lambda w(t))]>0$, $l=1,2$. It follows from (4.2) and (4.3) that

$$\bigl|p(t)z'(t)\bar{z}(t)\bigr|\geqslant \int_a^t \operatorname{Re}\bigl(e^{i\theta_l}p(\tau)\bigr)\bigl|z'(\tau)\bigr|^2\,d\tau,\quad l=1,2. \tag{4.5}$$

In addition, since $\theta_1\neq\theta_2 \pmod{\pi}$, it can be obtained from the proof of Theorem 2.1 that (2.10) holds. Hence, we get from (2.10) and (4.5) that

$$\bigl|p(t)z'(t)\bar{z}(t)\bigr|\geqslant \frac{1}{2M_0}V(t), \tag{4.6}$$

where

$$V(t)=\int_{t_1}^t \bigl|p(\tau)\bigr|\bigl|z'(\tau)\bigr|^2\,d\tau.$$

Now, we show that $z\notin L^2_w$. Suppose on the contrary that $z\in L^2_w$. Then it follows from (4.4) and Schwarz inequality that

$$V(t)=\int_{t_1}^t \frac{w(\tau)|p(\tau)z'(\tau)\bar{z}(\tau)|^2}{w(\tau)|p(\tau)||z(\tau)|^2}\,d\tau\geqslant h^2\|z\|^{-1}\left\{\int_{t_1}^t\sqrt{\frac{w(\tau)}{|p(\tau)|}}\,d\tau\right\}^2,$$

which yields that $\lim_{t\to b}V(t)=+\infty$ by the assumption. On the other hand, we get from (4.6) that

$$V'(t)=\bigl|p(t)\bigr|\bigl|z'(t)\bigr|^2=\frac{|p(t)z'(t)\bar{z}(t)|^2}{|p(t)||z(t)|^2}\geqslant\frac{1}{4M_0^2}\frac{V^2(t)}{|p(t)||z(t)|^2}.$$

So, it follows that

$$-\left(\frac{1}{V(t)}\right)'=\frac{V'(t)}{V^2(t)}\geqslant\frac{1}{4M_0^2}\frac{w(t)}{w(t)|p(t)||z(t)|^2}. \tag{4.7}$$

Integrating (4.7) from t_1 to t and using Schwarz inequality, we get that

$$\frac{1}{V(t_1)}-\frac{1}{V(t)}\geqslant\frac{1}{4M_0^2}\int_{t_1}^t\frac{w(\tau)}{w(\tau)|p(\tau)||z(\tau)|^2}\,d\tau\geqslant\frac{1}{4M_0^2}\|z\|^{-1}\int_{t_1}^t\sqrt{\frac{w(\tau)}{|p(\tau)|}}\,d\tau,$$

which implies that $\lim_{t\to b}\frac{1}{V(t)}=-\infty$. Then we have a contradiction with $\lim_{t\to b}V(t)=+\infty$. So, $z\notin L^2_w$ and Eq. (1.1) is in the case I by (i) of Remark 2.2. This completes the proof. □

Corollary 4.1. *Suppose that $p(t)\equiv 1$, $q(t)$ is real, $\int_a^b\sqrt{w(t)}\,dt=\infty$, and $\frac{q(t)}{w(t)}$ is bounded from below. Then Eq.* (1.1) *is in the case* I.

Proof. From the assumptions, we have

$$\Omega=\overline{\mathrm{co}}\left\{\frac{q(t)}{w(t)}+r,\ t\in[a,b),\ 0<r<\infty\right\}\neq\mathbb{C}.$$

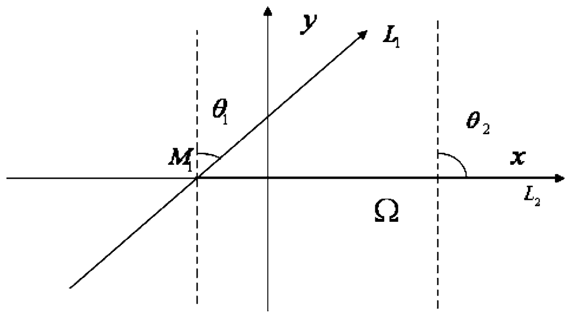

Fig. 1. The set Ω and the angles of rotation θ_1 and θ_2.

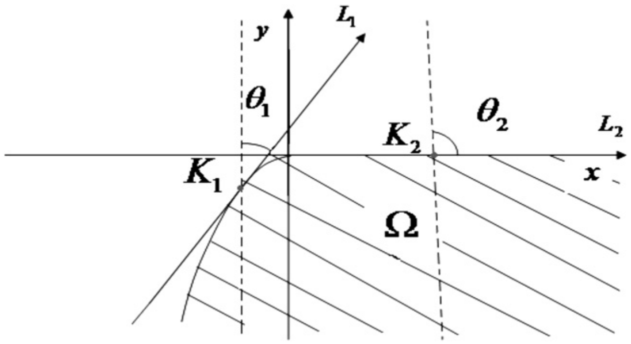

Fig. 2. The set Ω and the angles of rotation θ_1 and θ_2 for Eq. (4.8).

Clearly, $\mathcal{A}$ contains at least two different elements θ_1 and θ_2 (mod π) since all the points of Ω form a half line $x \geqslant M_1 := \inf\{\frac{q(t)}{w(t)}\}$ in the complex plane xoy (see Fig. 1).

In addition, $\int_a^b \sqrt{\frac{w(t)}{|p(t)|}}\,\mathrm{d}t = \int_a^b \sqrt{w(t)}\,\mathrm{d}t = \infty$. So, Eq. (1.1) is in the case I by Theorem 4.1. This completes the proof. □

Remark 4.1. Under the assumptions in Corollary 4.1, Eq. (1.1) is in the l.p.c. at $t = b$ of the Weyl's classification by (iii) of Remark 2.2. So, Corollary 4.1 extends the relevant results for (1.1) with real coefficients (cf., e.g., [10, Theorem 10.1.4]).

Example 4.1. Consider the equation

$$-y'' + \left(-t^3 - it^6\right)y = \lambda y, \quad t \in [0, \infty). \tag{4.8}$$

In this case, $p(t) = w(t) \equiv 1$ and $q(t) = -t^3 - it^6$. Then

$$\Omega = \overline{\mathrm{co}}\left\{-t^3 - it^6 + r,\ t \in [0, \infty),\ 0 < r < \infty\right\} \neq \mathbb{C}.$$

Clearly, $\mathcal{A}$ contains at least two different elements θ_1 and θ_2 (mod π) (see Fig. 2).

Further, it is clear that $\int_0^\infty \sqrt{\frac{w(t)}{|p(t)|}}\,\mathrm{d}t = \infty$. So, (1.1) is in the case I by Theorem 4.1.

In the following, denote $p_1(t) := \mathrm{Re}\, p(t)$, $q_1(t) := \mathrm{Re}\, q(t)$, $p_2(t) := \mathrm{Im}\, p(t)$, and $q_2(t) := \mathrm{Im}\, q(t)$. It is noted that Theorem 4.1 cannot be used for Eq. (1.1) for which there is only one element (mod π) in $\mathcal{A}$. So, we establish the following criterion which can be also used for this case.

Theorem 4.2. *Assume that there exist $a \leqslant t_0 < b$, four positive constants k_0, k_1, k_2, k_3, a positive absolutely continuous function $M(t)$ such that for $t \in [t_0, b)$, $p_1(t) > 0$, $M(t) \geqslant k_0$, $p_1(t) \geqslant k_1|p_2(t)|$, $q_1(t) \geqslant -k_2 M(t) w(t)$, $|p(t)|^{\frac{1}{2}} M'(t) M^{-\frac{3}{2}}(t) \leqslant k_3 w(t)^{\frac{1}{2}}$,*

$$\int_{t_0}^{b} \frac{\sqrt{p_1(t) w(t)}}{|p(t)|\sqrt{M(t)}}\,\mathrm{d}t = +\infty, \tag{4.9}$$

then Eq. (1.1) is in the case I.

Proof. Let $(\theta, K) \in S$ and $\Lambda_{\theta,K}$ be the corresponding half plane. Choose $\lambda \in \Lambda_{\theta,K}$. Let $z(t)$ be the solution of (1.1) satisfying the initial value conditions (4.1). Then (4.2) and (4.3) hold with θ_l and δ_l replaced by θ and δ, where δ is the distance from λ to $\partial\Lambda_{\theta,K}$. Then it follows that there exist $\tilde{t}_1 \in (a,b)$ and a positive constant $\tilde{h}$ such that

$$\left|p(t)z'(t)\bar{z}(t)\right| \geqslant \tilde{h}, \quad \tilde{t}_1 \leqslant t < b. \tag{4.10}$$

On the other hand, from the fact that $z(t)$ is the solution of (1.1), we get that

$$-\frac{(p(t)z'(t))'\bar{z}(t)}{M(t)} + \frac{q(t)|z(t)|^2}{M(t)} = \lambda\frac{w(t)|z(t)|^2}{M(t)}.$$

It follows that

$$\left(\frac{p(t)z'(t)\bar{z}(t)}{M(t)}\right)' = \frac{p(t)|z'(t)|^2}{M(t)} - \frac{M'(t)p(t)z'(t)\bar{z}(t)}{M^2(t)} + \frac{q(t)|z(t)|^2}{M(t)} - \lambda\frac{w(t)|z(t)|^2}{M(t)}. \tag{4.11}$$

Integrating both sides of (4.11) from $t_2 := \max\{t_0, \tilde{t}_1\}$ to t and then taking the real part, we can get that

$$\begin{aligned}\operatorname{Re}\left(\frac{p(t)z'(t)\bar{z}(t)}{M(t)}\right) &= V(t) - \operatorname{Re}\left\{\int_{t_2}^{t} \frac{M'(\tau)p(\tau)z'(\tau)\bar{z}(\tau)}{M^2(\tau)}\,\mathrm{d}\tau\right\} + \int_{t_2}^{t} \frac{q_1(\tau)|z(\tau)|^2}{M(\tau)}\,\mathrm{d}\tau \\ &\quad - \operatorname{Re}(\lambda)\int_{t_2}^{t} \frac{w(\tau)|z(\tau)|^2}{M(\tau)}\,\mathrm{d}\tau + c_0,\end{aligned} \tag{4.12}$$

where

$$V(t) = \int_{t_2}^{t} \frac{p_1(\tau)|z'(\tau)|^2}{M(\tau)}\,\mathrm{d}\tau \quad \text{and} \quad c_0 = \operatorname{Re}\left(\frac{p(t_2)z'(t_2)\bar{z}(t_2)}{M(t_2)}\right).$$

Next, we show that $z \notin L^2_w$. Assume the contrary. Suppose that $z \in L^2_w$. Then, using the assumptions and Schwarz inequality, we have

$$\operatorname{Re}\left\{\int_{t_2}^{t} \frac{M'(\tau)p(\tau)z'(\tau)\bar{z}(\tau)}{M^2(\tau)}\,\mathrm{d}\tau\right\} \leqslant k_3\|z\|\left\{\int_{t_2}^{t} \frac{|p(\tau)||z'(\tau)|^2}{M(\tau)}\,\mathrm{d}\tau\right\}^{1/2} \leqslant k_3\sqrt{1+\frac{1}{k_1^2}}\|z\|V^{1/2}(t).$$

Hence, we get from the assumptions and (4.12) that

$$\operatorname{Re}\left(\frac{p(t)z'(t)\bar{z}(t)}{M(t)}\right) \geqslant V(t) - k_3\sqrt{1+\frac{1}{k_1^2}}\|z\|V^{1/2}(t) - \left(k_2 + \frac{|\operatorname{Re}(\lambda)|}{k_0}\right)\|z\| + c_0. \tag{4.13}$$

Furthermore, it can be obtained from (4.10), $p_1(t) > 0$, and Schwarz inequality that

$$V(t) = \int_{t_2}^{t} \frac{p_1(\tau)|p(\tau)z'(\tau)\bar{z}(\tau)|^2}{M(\tau)|p(\tau)z(\tau)|^2}\,\mathrm{d}\tau \geqslant \tilde{h}^2\int_{t_2}^{t} \frac{p_1(\tau)}{M(\tau)|p(\tau)z(\tau)|^2}\,\mathrm{d}\tau \geqslant \tilde{h}^2\left\{\int_{t_2}^{t} \frac{\sqrt{p_1(\tau)w(\tau)}}{|p(\tau)|\sqrt{M(\tau)}}\,\mathrm{d}\tau\right\}^2\|z\|^{-1},$$

which, together with (4.9), implies that $\lim_{t\to b} V(t) = +\infty$. It can be obtained from $M(t) > k_0$, $\lim_{t\to b} V(t) = +\infty$, and (4.13) that there exists $\tilde{t}_2 \in (t_2, b)$ such that

$$\operatorname{Re}\left(p(t)z'(t)\bar{z}(t)\right) \geqslant k_0\operatorname{Re}\left(\frac{p(t)z'(t)\bar{z}(t)}{M(t)}\right) \geqslant \frac{k_0}{2}V(t), \quad t \in [\tilde{t}_2, b).$$

It follows that for $t \in [\tilde{t}_2, b)$,

$$V'(t) = \frac{p_1(t)|z'(t)|^2}{M(t)} \geqslant \frac{p_1(t)[\operatorname{Re}(p(t)z'(t)\bar{z}(t))]^2}{M(t)|p(t)z(t)|^2} \geqslant \frac{k_0^2}{4}\frac{p_1(t)V^2(t)}{M(t)|p(t)z(t)|^2}.$$

So, we have

$$-\left(\frac{1}{V(t)}\right)' = \frac{V'(t)}{V^2(t)} \geqslant \frac{k_0^2}{4}\frac{p_1(t)}{M(t)|p(t)z(t)|^2}. \tag{4.14}$$

Integrating (4.14) from $\tilde{t}_2$ to t and using Schwarz inequality, we get that

$$\frac{1}{V(\tilde{t}_2)}-\frac{1}{V(t)}\geqslant\frac{k_0^2}{4}\int_{\tilde{t}_2}^{t}\frac{p_1(\tau)}{M(\tau)|p(\tau)z(\tau)|^2}\,d\tau\geqslant\frac{k_0^2}{4}\|z\|^{-1}\int_{\tilde{t}_2}^{t}\frac{\sqrt{p_1(\tau)w(\tau)}}{|p(\tau)|\sqrt{M(\tau)}}\,d\tau,$$

which implies that $\lim_{t\to b}\frac{1}{V(t)}=-\infty$ by (4.9). Then, we get a contradiction with $\lim_{t\to b}V(t)=+\infty$. So, $z\notin L_w^2$ and Eq. (1.1) is in the case I by (i) of Remark 2.2. This completes the proof. $\square$

Remark 4.2. Compared with Theorem 4.1, there is no limitation on the number of the elements of $\mathcal{A}$ in Theorem 4.2. It can be also used for determining Eq. (1.1) to be in the case I when $\mathcal{A}$ contains exactly one element (mod π). For example, consider the equation

$$-\big(ty'\big)'+(-t+i)y=\lambda y,\quad t\in[0,\infty). \tag{4.15}$$

In this case, $p(t)=t$, $q(t)=-t+i$, and $w(t)\equiv 1$. Then

$$\Omega=\overline{\mathrm{co}}\big\{-t+i+rt,\ t\in[0,\infty),\ 0<r<\infty\big\}\neq\mathbb{C}.$$

It can be easily verified that all the points of Ω form a horizontal line $y=1$ in the complex plane xoy. So, $\mathcal{A}$ contains exactly one element $\frac{\pi}{2}$ (mod π), and consequently, Theorem 4.1 cannot be used for this equation. However, by choosing $M(t)=t$, it can be verified that (4.15) is in the case I by Theorem 4.2. Conversely, Theorem 4.1 is not included in Theorem 4.2. For example, Theorem 4.2 cannot be used for Eq. (4.8) since there does not exist a positive absolutely continuous function $M(t)$ such that it satisfies all the assumptions corresponding to $M(t)$ in Theorem 4.2. In fact, for (4.8), suppose that there exist $t_0\in[0,\infty)$, a positive constant k_2, and a positive absolutely continuous function $M(t)$ such that for $t\in[t_0,\infty)$, $q_1(t)\geqslant -k_2M(t)w(t)$, i.e., $-t^3\geqslant -k_2M(t)$. Then it follows that $\sqrt{M(t)}\geqslant\frac{1}{\sqrt{k_2}}t^{\frac{3}{2}}$. So,

$$\int_{t_0}^{\infty}\frac{\sqrt{p_1(t)w(t)}}{|p(t)|\sqrt{M(t)}}\,dt=\int_{t_0}^{\infty}\frac{1}{\sqrt{M(t)}}\,dt<+\infty,$$

and consequently, the assumption (4.9) does not hold for (4.8).

It is noted that more limitations are imposed on $\operatorname{Re}p(t)$ and $\operatorname{Re}q(t)$ in Theorem 4.2. Integrating both sides of (4.11) and taking the imaginary part, we can get the following criterion of the case I, in which more limitations are imposed on $\operatorname{Im}p(t)$ and $\operatorname{Im}q(t)$.

Theorem 4.3. *Assume that there exist $a\leqslant t_0<b$, four positive constants k_0, k_1, k_2, k_3, a positive absolutely continuous function $M(t)$ such that for $t\in[t_0,b)$, $p_2(t)>0$, $M(t)\geqslant k_0$, $p_2(t)\geqslant k_1|p_1(t)|$, $q_2(t)\geqslant -k_2M(t)w(t)$, $|p(t)|^{\frac{1}{2}}M'(t)M^{-\frac{3}{2}}(t)\leqslant k_3w(t)^{\frac{1}{2}}$,*

$$\int_{t_0}^{b}\frac{\sqrt{p_2(t)w(t)}}{|p(t)|\sqrt{M(t)}}\,dt=+\infty,$$

then Eq. (1.1) *is in the case* I.

Letting $M(t)=t^2$, we can get the following result from Theorems 4.2 and 4.3.

Corollary 4.2. *The equation*

$$-y''+q(t)y=\lambda y,\quad t\in[0,\infty),$$

with $q_1(t)>-kt^2$, where k is a positive constant, is in the case I, *and the equation*

$$-iy''+q(t)y=\lambda y,\quad t\in[0,\infty),$$

with $q_2(t)>-kt^2$ is in the case I.

Finally, we give the following criteria of the three cases by using the relationship between the classifications of Eq. (1.1) and system (3.2) obtained in Theorem 3.1.

Theorem 4.4. *Let $\tilde{w}(t):=\max\{|q(t)|,w(t)\}$.*

(1) *If* $\int_a^b \tilde{w}(t)\,\mathrm{d}t < \infty$ *and* $\int_a^b |p(t)|^{-1}\,\mathrm{d}t < \infty$, *then Eq.* (1.1) *is in the case* III *w.r.t. all* $\Lambda_{\theta,K}$. *Conversely, if* $\mathcal{A}$ *contains at least two different elements* θ_1 *and* θ_2 (mod π) *and Eq.* (1.1) *is in the case* III *w.r.t.* Λ_{θ_l,K_l} $(l = 1, 2)$, *then* $\int_a^b \tilde{w}(t)\,\mathrm{d}t < \infty$ *and* $\int_a^b |p(t)|^{-1}\,\mathrm{d}t < \infty$;

(2) *Assume that the set* $\mathcal{A}$ *contains at least two different elements* (mod π). *If* $\int_a^b \tilde{w}(t)\,\mathrm{d}t < \infty$, $\int_a^b |p(t)|^{-1}\,\mathrm{d}t = \infty$, *and* $\int_a^b (\int_a^t |p(\tau)|^{-1}\,\mathrm{d}\tau)^2 \tilde{w}(t)\,\mathrm{d}t < \infty$, *then Eq.* (1.1) *is in the case* II *w.r.t. all* $\Lambda_{\theta,K}$ *except that there exists at most one* θ (mod π) *such that Eq.* (1.1) *is in the case* III *w.r.t. such an* $\Lambda_{\theta,K}$;

(3) *If* $\int_a^b w(t)\,\mathrm{d}t = \infty$ *and there exists a positive constant* k_0 *such that* $w(t) \geqslant k_0|q(t)|$, *then Eq.* (1.1) *is in the case* I.

Proof. We first show result (1). Suppose that $\int_a^b \tilde{w}(t)\,\mathrm{d}t < \infty$ and $\int_a^b |p(t)|^{-1}\,\mathrm{d}t < \infty$. Let $Y(t) = (y(t), v(t))^T$ and

$$W_1(t) := \operatorname{diag}\bigl(\tilde{w}(t), \bigl|p(t)\bigr|^{-1}\bigr).$$

It is clear that the system

$$JY'(t) = 0, \quad t \in [a, b), \tag{4.16}$$

has two linearly independent solutions $Y_1 = (1, 0)^T \in L^2_{W_1}$ and $Y_2 = (0, 1)^T \in L^2_{W_1}$. Since for every $\theta \in \mathcal{A}$, there exists a positive constant k such that

$$\bigl|c(t, \lambda, \theta)\bigr| \leqslant k\tilde{w}(t), \qquad \bigl|d(t, \theta)\bigr| \leqslant k\bigl|p(t)\bigr|^{-1}, \tag{4.17}$$

where $c(t, \lambda, \theta)$ and $d(t, \theta)$ are the same as that in system (3.2), we get that (3.2) with $\xi = 0$ has two linearly independent solutions in $L^2_{W_1}$ by Lemma 3.2. Note that there exists a positive constant $\tilde{k}$ such that $w_1(t, \lambda, \theta) \leqslant \tilde{k}\tilde{w}(t)$ and $w_2(t, \theta) \leqslant \tilde{k}|p(t)|^{-1}$, where w_1 and w_2 are the same as that in (3.1). Then $f \in L^2_W$ if $f \in L^2_{W_1}$. So, (3.2) with $\xi = 0$ has two linearly independent solutions in L^2_W, and consequently, (3.2) is in l.c.c. w.r.t. $\theta \in \mathcal{A}$. Hence, Eq. (1.1) is in the case III w.r.t. $\Lambda_{\theta,K}$ by Theorem 3.1.

Now, suppose that $\mathcal{A}$ contains at least two different elements θ_1 and θ_2 (mod π). Then, (2.10) and (2.11) hold. Suppose that Eq. (1.1) is in the case III w.r.t. Λ_{θ_l,K_l}, $l = 1, 2$, and let $(\theta, K) \in S$. Then Eq. (1.1) is in the case III w.r.t. $\Lambda_{\theta,K}$ by Theorem 2.1. Choose $\lambda \in \Lambda_{\theta,K}$ and z be a solution of Eq. (1.1). Then, with a similar argument, we get that (2.12) holds. It follows from (2.10)–(2.12) that (2.13) holds. Further, it follows from the second inequality of (2.13), $|q(t)| \leqslant |q(t) - \lambda w(t)| + |\lambda w(t)|$, and $z \in L^2_w$ that

$$\int_a^b \bigl|q(t)\bigr|\bigl|z(t)\bigr|^2\,\mathrm{d}t < \infty. \tag{4.18}$$

Let $Y(t) = (z(t), v(t))^T$ with $v(t) = -ie^{i\theta}p(t)z'(t)$. Then Y is a solution of (3.3) by Lemma 3.1. Further, we get from the first inequality of (2.13), (4.18), and $z \in L^2_w$ that $Y \in L^2_{W_1}$. Since z is an arbitrary solution of (1.1), we get from Lemma 3.1 that all the solutions of (3.3) are in $L^2_{W_1}$. So, it follows from (4.17), Corollary 3.1, and Lemma 3.2 that all the solutions of (4.16) are in $L^2_{W_1}$. Note that $Y_1 = (1, 0)^T$ and $Y_2 = (0, 1)^T$ are two solutions of (4.16). So, $Y_1, Y_2 \in L^2_{W_1}$. Thus, $\int_a^b \tilde{w}(t)\,\mathrm{d}t < \infty$ and $\int_a^b |p(t)|^{-1}\,\mathrm{d}t < \infty$. Result (1) is proved.

Next, we prove result (2). Let $W_2(t) = \operatorname{diag}(\tilde{w}(t), 0)$. For $\theta \in \mathcal{A}$, the system

$$J\begin{pmatrix} y' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & d(t, \theta) + iw_2(t, \theta) \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}, \quad t \in [a, b), \tag{4.19}$$

has two linearly independent solutions $Y_1 = (1, 0)^T$, $Y_2 = (ie^{-i\theta}\int_a^t p^{-1}(\tau)\,\mathrm{d}\tau, 1)^T$. It holds that $Y_1, Y_2 \in L^2_{W_2}$ by the assumptions in (2) of this theorem. Since for every $\theta \in \mathcal{A}$, there exists a positive constant $\hat{k}$ such that

$$\bigl|c(t, \lambda, \theta)\bigr| \leqslant \hat{k}\tilde{w}(t), \qquad \bigl|w_1(t, \lambda, \theta)\bigr| \leqslant \hat{k}\tilde{w}(t), \tag{4.20}$$

we get form Lemma 3.2 that the system

$$J\begin{pmatrix} y' \\ v' \end{pmatrix} = \begin{pmatrix} -c(t, \lambda, \theta) + iw_1(t, \lambda, \theta) & 0 \\ 0 & d(t, \theta) + iw_2(t, \theta) \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}, \quad t \in [a, b), \tag{4.21}$$

has two linearly independent solutions, denoted by $Y_1 = (y_1, v_1)^T$ and $Y_1 = (y_2, v_2)^T$, in $L^2_{W_2}$. It is clear that $y_1, y_2 \in L^2_w$ since $Y_1, Y_2 \in L^2_{W_2}$ and $w(t) \leqslant \tilde{w}(t)$. Note that system (4.21) is exactly the system (3.3). Then, it follows that Eq. (1.1) has two linearly independent solutions in L^2_w by Lemma 3.1. So, Eq. (1.1) is not in the case I. Furthermore, if there exist two different elements θ_1 and θ_2 (mod π) such that Eq. (1.1) is in the case III w.r.t. Λ_{θ_l,K_l}, $l = 1, 2$, then $\int_a^b |p(t)|^{-1}\,\mathrm{d}t < \infty$

by result (1) of this theorem. It contradicts $\int_a^b |p(t)|^{-1}\,dt = \infty$. So, the conclusion (2) of this theorem holds. Result (2) is proved.

Finally, we prove result (3). Note that $w(t) \geqslant k_0|q(t)|$. So, $L^2_{\tilde{w}} = L^2_w$. Suppose that $\int_a^b w(t)\,dt = \infty$. Then the solution $Y_1 = (1,0)^T$ of (4.19) is not in $L^2_{W_2}$. It can be obtained from (4.20) and Lemma 3.2 that system (4.21), i.e., system (3.3), has a solution, denoted by $Y = (y, v)^T$, not to be in $L^2_{W_2}$. So, $y \notin L^2_w$. It follows that (1.1) has a solution not to be in L^2_w by Lemma 3.1. So, Eq. (1.1) is in the case I by (i) of Remark 2.2 and result (3) is proved. The entire proof is completed. □

Remark 4.3. From result (1) of Theorem 4.4 and Theorem 2.1, we get that if $\mathcal{A}$ contains at least two different elements θ_1 and θ_2 (mod π), Eq. (1.1) is in the case III w.r.t. all $\Lambda_{\theta,K}$ if and only if the coefficients of Eq. (1.1) satisfy the regularity conditions:

$$\int_a^b \tilde{w}(t)\,dt < \infty, \qquad \int_a^b \big|p(t)\big|^{-1}\,dt < \infty.$$

According to A. Zettl's classification of regular and singular endpoints [21, Capther 2], the above conditions implies that b is a regular endpoint, and hence result (1) of Theorem 4.4 states that: if $\mathcal{A}$ has more than one elements (mod π), then Eq. (1.1) is in the case III w.r.t. all $\Lambda_{\theta,K}$ if and only if the endpoint b is regular in the sense of A. Zettl's classification.

However, if $\mathcal{A}$ contains exactly one element (mod π), the coefficients of Eq. (1.1) may not satisfy the regularity conditions although Eq. (1.1) is in the case III w.r.t. all $\Lambda_{\theta,K}$, that is to say, under the condition that (1.1) is in the case III w.r.t. all $\Lambda_{\theta,K}$, the number of elements of $\mathcal{A}$ plays an important role in determining the regularity of the coefficients of Eq. (1.1). For example, consider the equation

$$-\big(e^{-t}y'\big)' - \big(e^{-t} + ie^{-3t}\big)y = \lambda e^{-3t}y, \quad t \in [0,\infty). \tag{4.22}$$

For Eq. (4.22), $p(t) = e^{-t}$, $q(t) = -e^{-t} - ie^{-3t}$, $w(t) = e^{-3t}$, and

$$\Omega = \overline{\mathrm{co}}\big\{-e^{2t} - i + re^{-t},\ t \in [0,\infty),\ 0 < r < \infty\big\} \neq \mathbb{C}.$$

It can be easily verified that all the points of Ω form the horizontal line $y = -1$ in the complex plane xoy. So, $\mathcal{A}$ contains exactly one element $\frac{\pi}{2}$ (mod π). Let $(\frac{\pi}{2}, K) \in S$. Since $\mathrm{Re}(e^{i\theta}p(t)) = \mathrm{Re}(e^{i\frac{\pi}{2}}e^{-t}) = 0$ and

$$\mathrm{Re}\big[e^{i\theta}\big(q(t) - Kw(t)\big)\big] = \mathrm{Re}\big[e^{i\frac{\pi}{2}}\big(-e^{-t} - (i+K)e^{-3t}\big)\big] = \big[1 + \mathrm{Im}(K)\big]w(t),$$

we get that (4.22) is in the case III w.r.t. $\Lambda_{\frac{\pi}{2},K}$ if all the solutions of (4.22) are in L^2_w. Further, (4.22) can be written as

$$-\big(e^{-t}y'\big)' - e^{-t}y = (\lambda + i)e^{-3t}y, \quad t \in [0,\infty), \tag{4.23}$$

with the spectral parameter $\lambda + i$. By (ii) of Remark 2.1, all the solutions of (4.23) are in L^2_w if and only if all the solutions of the equation

$$-\big(e^{-t}y'\big)' - e^{-t}y = 0, \quad t \in [0,\infty), \tag{4.24}$$

are in L^2_w. It can be obtained that $y_1 = e^{\frac{t}{2}}\cos\frac{\sqrt{3}t}{2}$ and $y_2 = e^{\frac{t}{2}}\sin\frac{\sqrt{3}t}{2}$ are two linearly independent solutions of (4.24). Clearly, $y_1, y_2 \in L^2_w$. So, all the solutions of (4.23) are in L^2_w. Hence, (4.22) is in the case III w.r.t. $\Lambda_{\frac{\pi}{2},K}$. However, $\int_0^\infty |p(t)|^{-1}\,dt = \int_0^\infty e^t\,dt = \infty$.

At the end of this present paper, we give two examples to indicate the applications of the results in this section.

Example 4.2. Consider the equation

$$-\big(p(t)y'\big)' - \mu\tilde{q}(t)y = \lambda w(t)y, \quad t \in [0,\infty), \tag{4.25}$$

where $p(t) > 0$ and $\tilde{q}(t) > 0$ for $t \in [0,\infty)$ and μ is a complex number. This example was constructed in [3]. For this equation, we have

$$\Omega = \overline{\mathrm{co}}\left\{-\mu\frac{\tilde{q}(t)}{w(t)} + rp(t),\ t \in [0,\infty),\ 0 < r < \infty\right\} \neq \mathbb{C}.$$

Let $\theta_0 = \frac{\pi}{2} - \arg\mu$. Then $\theta_0 \in \mathcal{A}$. Let K_0 be the corresponding point on $\partial\Omega$. By the discussions in [3, Section 4], it holds that

(1) if $p = w = \tilde{q} \equiv 1$, then (4.25) is in the case I;
(2) if $p \equiv 1$ and $w(t) = \tilde{q}(t) = (1+t^2)^{-2}$, then (4.25) is in the case II w.r.t. Λ_{θ_0,K_0};
(3) if $p(t) = (1+t^2)^2$, $w(t) = (1+t^2)^{-1}$, and $\tilde{q} \equiv 1$, then (4.25) is in the case III w.r.t. Λ_{θ_0,K_0}.

Here, we can conclude that (4.25) is in the case I under the assumptions in (1) of this example by arbitrary one of the three results: Corollary 4.1, Theorem 4.2, and (3) of Theorem 4.4. Further, it can be verified that $\mathcal{A}$ contains at least two different elements θ_1 and θ_2 (mod π). So, by (2) of Theorem 4.4, we can conclude that under the assumptions in (2) of this example, Eq. (4.25) is in the case II w.r.t. all $\Lambda_{\theta,K}$ except that there exists at most one θ (mod π) such that Eq. (4.25) is in the case III w.r.t. such an $\Lambda_{\theta,K}$. Note that all the solutions of (4.25) are in L^2_w and $\tilde{w}(t)=1$ satisfies $\int_a^b \tilde{w}(t)\,dt=\infty$ under the assumptions in (3) of this example. So, by (1) of Theorem 4.4, it can be concluded that under the assumptions in (3) of this example, Eq. (4.25) is in the case II w.r.t. all $\Lambda_{\theta,K}$ except for $\theta_0=\frac{\pi}{2}-\arg\mu$ (mod π).

Example 4.3. Consider the equation

$$-\big(\big(p_1t^{\alpha_1}+ip_2t^{\alpha_2}\big)y'\big)'+\big(q_1t^{\beta_1}+iq_2t^{\beta_2}\big)y=\lambda t^{\nu}y, \quad t\in[1,\infty), \tag{4.26}$$

where $p_j, q_j, \alpha_j, \beta_j$ $(j=1,2)$, and ν are real constants. In this case, $p(t)=p_1t^{\alpha_1}+ip_2t^{\alpha_2}$, $q(t)=q_1t^{\beta_1}+iq_2t^{\beta_2}$, $w(t)=t^{\nu}$. Let $p_j>0$ and $q_j>0$, $j=1,2$. Then

$$\Omega=\overline{\mathrm{co}}\big\{q_1t^{\beta_1-\nu}+iq_2t^{\beta_2-\nu}+r\big(p_1t^{\alpha_1}+ip_2t^{\alpha_2}\big),\ t\in[1,\infty),\ 0<r<\infty\big\}\neq\mathbb{C} \tag{4.27}$$

and there must exist two different elements (mod π) in $\mathcal{A}$ since all the points of Ω in (4.27) lie only in the first quadrant of the complex plane xoy. Let $A=\max\{\alpha_1,\alpha_2\}$, $B=\max\{\beta_1,\beta_2,\nu\}$, and $C=\max\{\beta_1,\beta_2\}$. Then we have

(1) if $A>1$ and $B<-1$, then it can be verified by result (1) of Theorem 4.4 that (4.26) is in the case III w.r.t. all $\Lambda_{\theta,K}$;
(2) if $\nu\geqslant -1$ and $\nu\geqslant C$, then it can be verified by result (3) of Theorem 4.4 that (4.26) is in the case I;
(3) if $A\leqslant 1$, $B<-1$, and $2A-B>3$, then Eq. (4.26) is in the case II w.r.t. all $\Lambda_{\theta,K}$ except that there exists at most one θ (mod π) such that Eq. (4.26) is in the case III w.r.t. such an $\Lambda_{\theta,K}$. In fact, it follows from $A\leqslant 1$ and $B<-1$ that $\int_1^\infty |p(t)|^{-1}\,dt=\infty$ and $\int_1^\infty \tilde{w}(t)\,dt<\infty$, where $\tilde{w}$ is the same as that in Theorem 4.4, and it follows from $2A-B>3$ that there exists $M_2>0$ such that

$$\int_1^\infty\left(\int_1^t \big|p(\tau)\big|^{-1}\,d\tau\right)^2\tilde{w}(t)\,dt\leqslant M_2\int_1^\infty\left(\int_1^t \tau^{-A}\,d\tau\right)^2 t^B\,dt=\frac{M_2}{(1-A)^2}\int_1^\infty\big(t^{1-A}-1\big)^2t^B\,dt$$
$$\leqslant\frac{2M_2}{(1-A)^2}\int_1^\infty t^{2(1-A)+B}\,dt<\infty.$$

Consequently, this conclusion holds by result (2) of Theorem 4.4.

Remark 4.4. By the asymptotic behavior of solutions of Eq. (1.1) with polynomial coefficients, one can identify the case of each of them, e.g., Eq. (4.26) [4]. It is noted that it is easy to identify the case of Eq. (1.1) with more general coefficients by the criteria established in the present paper.

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